Exact superpotentials for theories with flavors via a matrix integral

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We extend and test the method of Dijkgraaf and Vafa for computing the superpotential of $\mathcal{N}=1$ theories to include flavors in the fundamental representation of the gauge group. This amounts to computing the contribution to the superpotential from surfaces with one boundary in the matrix integral. We compute exactly the effective superpotential for the case of the gauge group $U(N_c)$, N_f massive flavor chiral multiplets in the fundamental and one massive chiral multiplet in the adjoint, together with a Yukawa coupling. We compare up to sixth order with the result obtained by standard field theory techniques in the already nontrivial case of $N_c=2$ and $N_f=1$. The agreement is perfect.

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In a recent paper, Dijkgraaf and Vafa [1] (building on previous work [2]) have proposed a simple technique for computing the effective superpotential for the glueball field $S=-(1/32\pi^2)\mathrm{tr}\, W^\alpha W_\alpha$ in a large class of $\mathcal{N}=1$ theories. For instance, in the case of a $U(N_c)$ gauge group and chiral fields Φ_i in the adjoint representation interacting with a tree level superpotential $W_{tree}(\Phi_i,\lambda_a)$, one is instructed to compute the matrix integral to leading order in N_c:

$$e^{-(N_c^2/S^2)\mathcal{F}_{\chi=2}(S,\lambda_a)}\approx\int d\Phi_i e^{-(N_c/S)W_{tree}(\Phi_i,\lambda_a)}, \qquad (1)$$

where we have denoted by λ_a the coupling constants appearing in the superpotential. The effective superpotential is, in this case [1],

$$W_{DV}(S,\Lambda,\lambda_a)=N_cS[-\log(S/\Lambda^3)+1]+N_c\frac{\partial\mathcal{F}_{\chi=2}(S,\lambda_a)}{\partial S}, \qquad (2)$$

where the presence on $N_c(\partial/\partial S)$ is justified by the combinatorics of diagrams written on surfaces with spherical topology. The first piece of the superpotential is the Veneziano-Yankielowicz superpotential for pure $SU(N_c)$ Super Yang-Mills (SYM) theory [3], while the second piece which starts with $O(S^2)$ terms gives the instanton corrections. In the case that the matrix model is integrable we can write the exact effective superpotential in closed form; otherwise we can compute it at any given order in S. Recent checks and developments of the conjecture have been performed in [4–12].

In the case of gauge groups $SO(N_c)$ or $Sp(N_c)$ there are also contributions from nonorientable diagrams that can be written on the projective plane and their contribution $G_{\chi=1}(S,\lambda_a)$ will appear in Eq. (2) without the factor $N_c(\partial/\partial S)$.

It is a natural step to extend the conjecture to theories including matter, that is N_f chiral multiplets in the fundamental. This is implemented simply by including surfaces with boundaries. To be specific, in the case of gauge group $U(N_c)$, for adjoint matter Φ_i and fundamental matter Q_f and $\tilde{Q}^f$ one should first compute

$$e^{-(N_c^2/S^2)\mathcal{F}_{\chi=2}(S,\lambda_a)-(N_c/S)\mathcal{F}_{\chi=1}(S,\lambda_a)}\approx\int d\Phi_i dQ_f d\tilde{Q}^f e^{-(N_c/S)W_{tree}(\Phi_i,Q_f,\tilde{Q}^f,\lambda_a)}, \qquad (3)$$

and then write [the non-orientable contribution $G_{\chi=1}(S,\lambda_a)$ is absent in this case]

$$W_{DV}(S,\Lambda,\lambda_a)=N_cS[-\log(S/\Lambda^3)+1]+N_c\frac{\partial\mathcal{F}_{\chi=2}(S,\lambda_a)}{\partial S}+\mathcal{F}_{\chi=1}(S,\lambda_a). \qquad (4)$$

We are now going to test this conjecture. We take a $U(N_c)$ gauge theory with one adjoint chiral multiplet Φ and N_f chiral multiplets in the fundamental Q_f and $\tilde{Q}^f$. The tree level superpotential gives masses to all matter fields and, moreover, there is a cubic coupling between the fundamentals and the adjoint. All other possible couplings are turned off. The tree level superpotential reads

$$W_{tree}=\frac{1}{2}M\,\mathrm{tr}\,\Phi^2+mQ_f\tilde{Q}^f+gQ_f\Phi\tilde{Q}^f, \qquad (5)$$

where the flavor indices are summed while the color indices are not written explicitly.

Since there are no self-interactions of the adjoint field Φ, all diagrams with interactions will involve at least one flavor loop, that is a boundary. The genus zero piece of the matrix integral reduces trivially to the (one loop) vacuum amplitude of the adjoint field, which enforces the matching of scales in the Veneziano-Yankielowicz piece of the superpotential. A similar factor is also present in the flavor integral with analogous consequences. By working directly with the scale of the pure SYM theory we can concentrate on the interacting part which receives no contributions from genus zero. To leading order in N_c, the matrix integral is thus saturated by planar diagrams with one boundary, which sum up to $\mathcal{F}_{\chi=1}$.

The matrix integration can thus be easily performed. Write

$$Z=\int d\Phi dQ_f d\tilde{Q}^f e^{-(N_c/S)[(1/2)M\,\mathrm{tr}\,\Phi^2+mQ_f\tilde{Q}^f+gQ_f\Phi\tilde{Q}^f]}$$
$$=\langle e^{-(N_c/S)gQ_f\Phi\tilde{Q}^f}\rangle, \tag{6}$$

where the correlators are normalized such that

$$\langle Q_{\alpha f}\tilde{Q}^{\beta g}\rangle=\frac{1}{m}\frac{S}{N_c}\delta_\alpha^\beta\delta_f^g,\quad \langle\Phi_\beta^\alpha\Phi_\lambda^\gamma\rangle=\frac{1}{M}\frac{S}{N_c}\delta_\lambda^\alpha\delta_\beta^\gamma. \tag{7}$$

Expanding the exponential we have

$$\langle e^{-(N_c/S)gQ_f\Phi\tilde{Q}^f}\rangle$$
$$=\sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(\frac{gN_c}{S}\right)^{2k}\langle(Q_f\Phi\tilde{Q}^f)_1(Q_f\Phi\tilde{Q}^f)_2\cdots$$
$$\times(Q_f\Phi\tilde{Q}^f)_{2k}\rangle, \tag{8}$$

where we took into account that only correlators of an even number of fields Φ are nonzero.

It is a simple combinatorial exercise to extract from Eq. (8) the coefficients of the connected planar diagrams with one boundary. The different diagrams can be obtained first by contracting the Qs and $\tilde{Q}$s in $(2k-1)!$ ways to give a single boundary, and then connecting $2k$ points on the boundary through k nonintersecting lines (the $\langle\Phi\Phi\rangle$ propagators). The solution to this last combinatorial problem can be found in Eq. (31) of [13]. The result for the free energy is

$$\frac{N_c}{S}\mathcal{F}_{\chi=1}(S,g,m,M)=-N_f\sum_{k=1}^{\infty}\frac{(2k-1)!}{(k+1)!k!}\left(\frac{gN_c}{S}\right)^{2k}$$
$$\times\left(\frac{S}{mN_c}\right)^{2k}\left(\frac{S}{MN_c}\right)^{k}N_c^{k+1}, \tag{9}$$

which we can rewrite, for $\alpha=g^2/m^2M$, as

$$\mathcal{F}_{\chi=1}(S,\alpha)=-N_f\sum_{k=1}^{\infty}\frac{(2k-1)!}{(k+1)!k!}\alpha^kS^{k+1}. \tag{10}$$

This expression can actually be summed to give

$$\mathcal{F}_{\chi=1}(S,\alpha)=-N_fS\left[\frac{1}{2}+\frac{1}{4\alpha S}(\sqrt{1-4\alpha S}-1)\right.$$
$$\left.-\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\alpha S}\right)\right]. \tag{11}$$

We thus claim that

$$W_{DV}=N_cS[-\log(S/\Lambda^3)+1]+\mathcal{F}_{\chi=1}(S,\alpha) \tag{12}$$

is the exact superpotential for our theory with N_f flavors and the Yukawa coupling to the adjoint matter field.

Our next task is to provide a purely field theory deduction of the effective potential, and check that it matches the expression computed through the matrix model. This task can be performed by considering the system as an $\mathcal{N}=1$ deformation of an $\mathcal{N}=2$ SYM theory with matter, and then deducing the low energy superpotential through the Seiberg-Witten curve of the system. While this is an interesting problem, it is already possible to prove the power of the DV approach in the simplest case, that is $N_c=2$ and $N_f=1$. This case already gives a nontrivial result from the matrix model, and we are going to show that it matches precisely the field theory exact superpotential that we can simply obtain by the standard techniques of [14,15]. In turn, this is a strong support for the conjectured exact superpotential (12).

Note that the superpotential (12) does not seem to discern between $N_f<N_c$ and $N_f\geqslant N_c$ while, for instance, the Seiberg-Witten curve of the related systems does. A hint that our solution for the matrix integral could break down is that for $N_f\geqslant N_c$, additional boundaries start giving a large contribution to the integral, so that the $\chi=1$ term is no longer singled out.

Let us write the first few terms of our exact superpotential in the case $N_c=2$ and $N_f=1$:

$$W_{DV}=2S[-\log(S/\Lambda^3)+1]-\frac{1}{2}\alpha S^2-\frac{1}{2}\alpha^2S^3-\frac{5}{6}\alpha^3S^4$$
$$-\frac{7}{4}\alpha^4S^5-\frac{21}{5}\alpha^5S^6-11\alpha^6S^7+O(\alpha^7). \tag{13}$$

Now we integrate S out by setting $\partial_SW_{DV}=0$ and solve for $S(\Lambda)$ as an expansion in α. Plugging back into Eq. (13) we find

$$W_{DV}(\Lambda,\alpha)=2\Lambda^3\left[1-\frac{1}{4}\alpha\Lambda^3-\frac{1}{8}(\alpha\Lambda^3)^2-\frac{1}{8}(\alpha\Lambda^3)^3\right.$$
$$-\frac{21}{128}(\alpha\Lambda^3)^4-\frac{1}{4}(\alpha\Lambda^3)^5-\frac{429}{1024}(\alpha\Lambda^3)^6$$
$$\left.+O(\alpha^7)\right]. \tag{14}$$

We can now set out to obtain the same effective superpotential through an independent route. Consider the $U(2)$ theory with an adjoint matter field Φ and $N_f=1$ chiral fields in the fundamental Q and $\tilde{Q}$. Moreover, let us denote by $\tilde{\Lambda}$ the scale of this theory [to be more precise, of the $SU(2)$ factor]. The tree level superpotential we introduce is

$$W_{tree}=\frac{1}{2}M\,\mathrm{tr}\,\Phi^2+mQ\tilde{Q}+gQ\Phi\tilde{Q}. \tag{15}$$

Let us first integrate out the adjoint field Φ. This trivially gives $\Phi=-(g/M)\tilde{Q}Q$, and substituting into Eq. (15) we get, in the notation of [15]

$$W_{tree,d}=mX-\frac{1}{2}\frac{g^2}{M}X^2, \tag{16}$$

where $X=Q\tilde{Q}$ is the gauge invariant meson field.

We now add the Affleck-Dine-Seiberg piece to the superpotential [16], taking into account the matching of the scales

$\hat{\Lambda}^5 = M^2 \tilde{\Lambda}^3$, where $\hat{\Lambda}$ is the scale of the gauge theory with one flavor. The exact effective superpotential is thus given by

$$W_{eff} = \frac{\hat{\Lambda}^5}{X} + mX - \frac{1}{2}\frac{g^2}{M}X^2. \tag{17}$$

The absence of further corrections can be checked with the fact that the above superpotential leads to the right Seiberg-Witten curve, see for instance [17]. We can now integrate out the massive meson X and thus obtain the effective superpotential for the low energy pure $SU(2)$ theory [the $U(1)$ factor is now decoupled], whose scale Λ is matched to be $\Lambda^6 = m\hat{\Lambda}^5$. The scale Λ is now the same as in Eq. (14).

The condition for the extremum of W_{eff} is

$$ax^3 - x^2 + 1 = 0, \tag{18}$$

where $a \equiv (g^2/m^2M)\Lambda^3 = \alpha\Lambda^3$ and $x = (m/\Lambda^3)X$. The solution, expanded in powers of a, is

$$x = 1 + \frac{1}{2}a + \frac{5}{8}a^2 + a^3 + \frac{231}{128}a^4 + \frac{7}{2}a^5 + \frac{7293}{1024}a^6 + O(a^7). \tag{19}$$

Plugging Eq. (19) back into the effective superpotential (17), we get

$$W_{eff} = 2\Lambda^3\left(1 - \frac{1}{4}a - \frac{1}{8}a^2 - \frac{1}{8}a^3 - \frac{21}{128}a^4 - \frac{1}{4}a^5 - \frac{429}{1024}a^6 + O(a^7)\right), \tag{20}$$

which agrees exactly with Eq. (14).

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[1] R. Dijkgraaf and C. Vafa, ''A Perturbative Window into Non-Perturbative Physics,'' hep-th/0208048.
[2] R. Dijkgraaf and C. Vafa, Nucl. Phys. **B644**, 3 (2002); **B644**, 21 (2002); R. Gopakumar and C. Vafa, Adv. Theor. Math. Phys. **3**, 1415 (1999); C. Vafa, J. Math. Phys. **42**, 2798 (2001); F. Cachazo, K.A. Intriligator, and C. Vafa, Nucl. Phys. **B603**, 3 (2001).
[3] G. Veneziano and S. Yankielowicz, Phys. Lett. **113B**, 231 (1982).
[4] L. Chekhov and A. Mironov, Phys. Lett. B **552**, 293 (2003).
[5] N. Dorey, T.J. Hollowood, S. Prem Kumar, and A. Sinkovics, J. High Energy Phys. **11**, 039 (2002); **11**, 040 (2002).
[6] M. Aganagic and C. Vafa, ''Perturbative derivation of mirror symmetry,'' hep-th/0209138.
[7] F. Ferrari, Nucl. Phys. **B648**, 161 (2003).
[8] H. Fuji and Y. Ookouchi, J. High Energy Phys. **12**, 067 (2002).
[9] D. Berenstein, Phys. Lett. B **552**, 255 (2003).
[10] N. Dorey, T.J. Hollowood, and S.P. Kumar, J. High Energy Phys. **12**, 003 (2002).
[11] R. Dijkgraaf, S. Gukov, V.A. Kazakov, and C. Vafa, ''Perturbative Analysis of Gauged Matrix Models,'' hep-th/0210238.
[12] A. Gorsky, Phys. Lett. B **554**, 185 (2003).
[13] E. Brezin, C. Itzykson, G. Parisi, and J.B. Zuber, Commun. Math. Phys. **59**, 35 (1978).
[14] K.A. Intriligator, R.G. Leigh, and N. Seiberg, Phys. Rev. D **50**, 1092 (1994).
[15] K.A. Intriligator, Phys. Lett. B **336**, 409 (1994).
[16] I. Affleck, M. Dine, and N. Seiberg, Nucl. Phys. **B241**, 493 (1984); **B256**, 557 (1985).
[17] K.A. Intriligator and N. Seiberg, Nucl. Phys. **B431**, 551 (1994).